On the Spectral Gap for Convex Domains

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Abstract

Let D be a convex planar domain, symmetric about both the x- and y-axes, which is strictly contained in $(-a,a)\times(-b,b)=\Gamma$. It is proved that, unless D is a certain kind of rectangle, the difference (gap) between the first two eigenvalues of the Dirichlet Laplacian in D is strictly larger than the gap for Γ. We show how to give explicit lower bounds for the difference of the gaps.

1 Introduction

Let $0<\lambda_1^\Omega<\lambda_2^\Omega$ be the first two eigenvalues of the Dirichlet Laplacian for the bounded planar domain Ω. This paper is concerned with the spectral gap $\lambda_2^\Omega-\lambda_1^\Omega$ of Ω. The gap is the rate at which the Dirichlet heat kernel $p_t^\Omega(x,\cdot)$, normalized to have integral one, converges to the first eigenfunction, also normalized to have integral one, where convergence here can mean L^1 convergence, L^2 convergence, or pointwise convergence. We note the paper [15] says the gap "needs no motivation." Other papers concerned with gaps of convex planar domains include [2, 3, 5, 9, 13, 15, 16, 17, 18].

The gap of $(-a,a)\times(-b,b)=\Gamma$ is $3\pi^2/4\max(a,b)^2$. Davis proved in [9] that if D is doubly symmetric and convex, and contained in Γ, the gap of D is no smaller than the gap of Γ. Neither the proof of this in [9] nor subsequent proofs

(see [3], [2], and [11]) give results about strict inequality. Now $(-c,c)\times(-b,b)$ is strictly contained in Γ if $c<a$, but if $b\geq a$ it has the same gap, $3\pi^2/4b^2$, as Γ. We show such rectangles are the only exceptions.

Theorem 1 *If D is convex and symmetric about both the x- and y-axes and strictly contained in $(-a,a)\times(-b,b)$, and is not a rectangle of the form $(-c,c)\times(-b,b)$ or $(-a,a)\times(-c,c)$, then the gap of D exceeds the gap of $(-a,a)\times(-b,b)$.*

In common with previous work in [3], [11], and [9], our proof of Theorem 1 uses the consequence of a theorem of Payne [14] that for D as in Theorem 1, either the intersection of the x-axis with D or the intersection of the y-axis with D is a nodal line for a second eigenfunction. Thus a second eigenfunction of D is the first eigenfunction of either the right or the top half of D. The first eigenvalue is the rate of decay of the heat kernel, and so the proof given in the next section, that the following proposition implies Theorem 1, is quick. For any set $A\subset\mathbb{R}^2$, let $A^+=\{(x,y)\in A \mid x>0\}$.

Proposition 2 *Let D be a convex domain and symmetric about both the x- and the y-axes such that $(a,0)$, and $(0,b)$ are boundary points of D. Suppose $\Gamma=(-a,a)\times(-b,b)$ strictly contains D. Then if $z_0=(x_0,y_0)\in D^+$,*

$$\frac{\int_{D^+} p_t^{D^+}(z_0,z)\,dz}{\int_D p_t^D(z_0,z)\,dz}=o(1)\frac{\int_{\Gamma^+} p_t^{\Gamma^+}(z_0,z)\,dz}{\int_\Gamma p_t^\Gamma(z_0,z)\,dz}\qquad \text{as } t\to\infty. \tag{1}$$

It is not hard to show that Proposition 2 implies that $o(1)$ in fact decreases exponentially as $t\to\infty$. Our proof yields estimates on $o(1)$ which depend on the shape of D and translate into estimates on the difference of the gaps of D and Γ. More precisely, we prove the following theorem. Let $\overline{A}$ be the closure of A, let D be as in Theorem 1, and let $\Theta_{a,b}=\Theta=\{(x,y)\mid 0<x<a, -\frac{b}{a}x+b\leq y<b\}$, and note that if D is as in Theorem 1 and contains the points $(a,0)$ and $(0,b)$ then any point in the first quadrant which is in $(-a,a)\times(-b,b)$ but not in D must be in Θ.

Theorem 3 *Let the domain $D\subsetneq(-a,a)\times(-b,b)$ be convex and symmetric about both the x- and y-axes. There is a computable positive function $g_{a,b}=g$ on Θ such that if D contains $(a,0)$ and $(0,b)$ but D does not contain $(u_0,v_0)\in\Theta$ then*

$$\lambda_2^D-\lambda_1^D\geq\frac{3\pi^2}{4\max(a,b)^2}+g(u_0,v_0).$$

We do not find the largest possible g, nor do we know how to do this. By a computable function we mean a function of a, b, u_0, and v_0, involving only elementary one-dimensional functions. The statement that results if "computable" is removed from the statement of Theorem 3 follows fairly quickly from Theorem 1. See the end of Section 3.

In Section 4 we discuss possible extensions of Proposition 2 in which convexity and symmetry around both the x- and y-axes are respectively replaced

by convexity in x (i.e., two points in D with the same y value can be joined with a line segment lying in D) and symmetry about only the y-axis. We also discuss possible extensions of our results to certain Schrödinger operators and to higher dimensions. These extensions would lead to inequalities for the difference between two first eigenvalues but not to inequalities for spectral gaps, since no analogs of Payne's theorem are known in these settings.

The first use of ratios involving heat kernels to bound gaps was in [9]. In [3] different proofs of the results of [9] were given as well as a number of interesting generalizations. (See Section 4 of this paper.) One of these was a ratio inequality involving integrals of heat kernels. These are easier to prove than pointwise inequalities for heat kernels and yield the same information about gaps, which is why we use ratios of integrals in (1).

It is easy to modify an example in [16] to show that if $H_\varepsilon = (-1,1)\times(-1,1)\setminus \{(0,y) \mid |y| \geq \varepsilon\}$, then the gap of H_ε goes to 0 as ε approaches 0. Thus without the convexity condition or something to replace it, the conclusion of Theorem 1 does not hold. This example also shows that Theorems 3.2, 3.3, and 3.4 of [11] are incorrect.

2 Proof of Proposition 2

The proof of Proposition 2 is based on the connection between the heat kernel and Brownian motion and the approximation of Brownian motion by random walks.

In this section we work only with bounded planar domains. Some of our formulas will hold for all such domains; for these we use Ω to designate a domain. Other formulas are not claimed to hold for all bounded domains but do hold for all bounded convex domains; for these we use D to designate a domain. We work only with first, ground state eigenfunctions of a domain Ω, and we use ϕ^Ω to denote this eigenfunction normalized to integrate to one. The corresponding eigenvalue is denoted by λ^Ω.

Standard one-dimensional Brownian motion is denoted by W_t, $t \geq 0$. We use subscripts to denote initial position, as in P_x and E_x, so for example $P_3(W_0 = 3) = 1$. Standard two-dimensional Brownian motion is denoted $Z_t = (X_t, Y_t)$, $t \geq 0$. We define $R_0, R_1, \ldots$ to be a random walk such that $\{R_i - R_{i-1}\}_{i\geq 1}$ are independent and satisfy $P(R_i - R_{i-1} = -1) = P(R_i - R_{i-1} = 0) = P(R_i - R_{i-1} = 1) = 1/3$. The process W^n is the following scaled version of this walk. Let $R_0 = 0$. Let $\theta_n = 3 \cdot 2^{2n-1}$ and note $\mathrm{Var}(2^{-n} R_{\theta_n}) = 1$. Let $\Theta(n) = \{k\theta_n^{-1} \mid k = 0,1,2,\ldots\}$. Then W^n, started at 0, is defined by $W^n_{k\theta_n^{-1}} = 2^{-n} R_k$ so $\mathrm{Var}\, W^n_t = t$, $t \in \Theta(n)$. For $t \geq 0$ not in $\Theta(n)$, define $W^n_t = W^n_{k\theta^{-n}}$ if $t \in (k\theta_n^{-1}, (k+1)\theta_n^{-1})$, $k = 0,1,2,\ldots$. Two dimensional scaled random walk is denoted $Z^n = (X^n, Y^n)$. For this walk, X^n and Y^n are independent and both have the distribution of W^n. We let $\tau_\Omega = \inf\{t > 0 \mid Z_t \notin \Omega\}$ and $\tau^n_\Omega = \inf\{t > 0 \mid Z^n_t \notin \Omega\}$, and if I is an interval, $\tau_I = \inf\{t > 0 \mid W_t \notin I\}$ and $\tau^n_I = \inf\{t > 0 \mid W^n_t \notin I\}$. We use $a_t \sim b_t$ to indicate $\lim_{t\to\infty} a_t/b_t \in (0,\infty)$.

The eigenfunction expansion of the heat kernel (see Theorem II.4.13 in [4]) implies

$$P_z\left(\tau_\Omega > t\right) = \int_\Omega p_t^\Omega(z,w)\,dw \sim e^{-\lambda^\Omega t}. \tag{2}$$

Theorem 1 follows easily from (2) and Proposition 2. For a set $A \subset \mathbb{R}^2$ let $A^T = A \cap \{y > 0\}$. Then Payne's theorem implies the minimum of $\lambda^{D^+} - \lambda^D$ and $\lambda^{D^T} - \lambda^D$ is the gap of D and the minimum of $\lambda^{\Gamma^+} - \lambda^\Gamma$ or $\lambda^{\Gamma^T} - \lambda^\Gamma$ is the gap of Γ. Now by (2),

$$\lim_{t\to\infty} e^{(\lambda^{D^+} - \lambda^D)t} \frac{\int_{D^+} p_t^{D^+}(z_0,z)\,dz}{\int_D p_t^D(z_0,z)\,dz} \in (0,\infty)$$

with a similar formula for Γ^+ and Γ. Thus Proposition 2 implies

$$\lambda^{D^+} - \lambda^D > \lambda^{\Gamma^+} - \lambda^\Gamma.$$

Rotating D and Γ by $90°$ and using Proposition 2 for these rotated sets gives

$$\lambda^{D^T} - \lambda^D > \lambda^{\Gamma^T} - \lambda^\Gamma,$$

and these two inequalities give Theorem 1.

We note that we can without loss of generality assume that the closure of the D of Theorem 1 contains $(a,0)$ and $(0,b)$, since if this is not the case we can replace $(-a,a)\times(-b,b)$ with the smallest oriented rectangle which contains D, which rectangle could not have a smaller gap. We also note that the proof of Proposition 2 is virtually identical for all a and b, and that the assumption that $(a,0)$ and $(0,b)$ are in $\overline{D}$ does not alter the proof. Thus we will prove Proposition 2 only under the assumptions $\Gamma = S := (-1,1)\times(-1,1)$ and both $(1,0)$ and $(0,1)$ belong to $\overline{D}$.

Convex planar domains are Lipschitz, and thus the following lemma, which states that the heat kernel for convex domains is intrinsically ultracontractive, is a consequence of the estimates of [12]. (See the tenth line from the bottom of page 618 of [12].) An essentially stronger result may be found in [1]. Intrinsic ultracontractivity was introduced by Davies and Simon in [8].

Lemma 4 *There is a positive increasing function $c^D(t)$ on $(0,\infty)$, and a decreasing function $C^D(t)$ on $(0,\infty)$, such that* $\lim_{t\to\infty} c^D(t) = \lim_{t\to\infty} C^D(t) = 1$, *and for all $x, y \in D$ and $t > 0$,*

$$\frac{c^D(t)\phi^D(x)\phi^D(y)e^{-\lambda^D t}}{\int_D \phi(x)^2\,dx} \le p_t^D(x,y) \le \frac{C^D(t)\phi^D(x)\phi^D(y)e^{-\lambda^D t}}{\int_D \phi(x)^2\,dx}. \tag{3}$$

Integrating (3) in y and using (2) gives

$$\frac{c^D(t)\phi^D(x)e^{-\lambda^D t}}{\int_D \phi(x)^2\,dx} \le P_x\left(\tau_D > t\right) \le \frac{C^D(t)\phi^D(x)e^{-\lambda^D t}}{\int_D \phi(x)^2\,dx}. \tag{4}$$

One dimensional versions of (3) and (4) where D is a finite open interval, follow from the references provided for (3). These one dimensional inequalities are also pretty easy to prove directly from the equations of either the heat kernel or eigenfunctions, which are known for intervals.

The next lemma is a consequence of the classical fact that given a two-dimensional Brownian motion Z_t, $t \geq 0$, there is a sequence of n-scaled random walks Z_t^n, such that for any $\varepsilon > 0$ and any $K > 0$,

$$P\left(\max_{0 \leq s \leq K} |Z_s - Z_s^n| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty. \tag{5}$$

A proof of this well known fact is sketched in [10]. Here we are abusing notation a little as Z^n in (5) stands for a specific n-scaled random walk while in the following lemma it stands for a generic n-scaled random walk.

Lemma 5 *Let Ω be a domain in $\mathbb{R}^2$ and let $Q_1, Q_2, \ldots, Q_m$ be convex subdomains of Ω. Then for any $0 < t_1 < \cdots < t_m$, and any $z \in \Omega$,*

$$\lim_{n\to\infty} P_z\left(Z_{t_i}^n \in Q_i, 1 \leq i \leq m, \tau_\Omega^n > t_m\right) = P_z\left(Z_{t_i} \in Q_i, 1 \leq i \leq m, \tau_\Omega > t_m\right). \tag{6}$$

Especially,

$$\lim_{n\to\infty} P_z\left(\tau_\Omega^n > t\right) = P_z\left(\tau_\Omega > t\right) \qquad t > 0. \tag{7}$$

Again, (6) and (7) are known. A proof of (7) is sketched in [10]. The equality (6) follows quickly from (5), the fact that the probability that Z_{t_i} belongs to the boundary of Q_i equals zero, the fact that the probability that Z_{t_m} belongs to the boundary of Ω equals 0, and the fact (see [10]) that the probability that Z hits the boundary of Ω for some $t < t_m$ but does not hit the complement of the closure of Ω for some $t < t_m$ equals zero.

We denote by L_t^Ω, $t \geq 0$, the Markov process which has transition probabilities

$$\begin{aligned} l_t^\Omega(x,y) &= \frac{p_t^\Omega(x,y)\phi^\Omega(y)}{\int_\Omega p_t^\Omega(x,z)\phi^\Omega(z)\,dz} \\ &= p_t^\Omega(x,y)\frac{\phi^\Omega(y)}{\phi^\Omega(x)}e^{\lambda^\Omega t}, \end{aligned} \tag{8}$$

and stationary distribution

$$\psi^\Omega(x) = \frac{\phi^\Omega(x)^2}{\int_\Omega \phi^\Omega(y)^2\,dy}.$$

(See Theorem II.4.13 in [4] for (8).) This process is often called Brownian motion in Ω conditioned to never exit Ω.

Let $s_1 < r < s_2$. The conditional distribution of Z_r, $s_1 < r < s_2$, given $Z_{s_1} = z$ and $Z_{s_2} = w$ and $\tau_\Omega > s_2$, is exactly the same as the conditional distribution of L_r^Ω, $s_1 < r < s_2$, given $L_{s_1}^\Omega = z$ and $L_{s_2}^\Omega = w$. This follows by

computing directly the joint densities of these two processes at points $r_1 < r_2 < \cdots < r_n$, where $s_1 < r_1$ and $s_2 > r_n$, a computation we omit despite the fact that it is pleasing to see the eigenfunctions in the equations for L cancel away.

Now let $s = s(t)$ be such that $0 < s < t$ and both s and $t - 2s$ approach infinity as t approaches infinity. Let $\alpha_x(z, w)$ be the joint density of (Z_s, Z_{t-s}) given $\tau_D > t$ and $Z_0 = x$. Let $\beta(z, w)$ be the joint density of (L_s^D, L_{t-s}^D) given that L_0^D has density ψ^D, its stationary density. So β depends on $t - s$ while α_x depends on x, s, and $t - s$. Note that $\alpha_x(z, w)$ is the normalization, to integrate to 1, of

$$p_s^D(x, z) p_{t-2s}^D(z, w) P_w\left(\tau_D > s\right),$$

while

$$\beta(z, w) = \psi^D(z) l_{t-2s}^D(z, w).$$

Therefore, (3) and (4) imply that $\alpha_x(z, w)/\beta(z, w)$ converges uniformly to 1 in $D \times D$, as $t \to \infty$, at a rate which may be taken independent of x.

Thus, if $\mathcal{B}[s, t - s]$ is the σ-field of the continuous functions on $[s, t - s]$ generated by the projection maps, and $x \in D^+$,

$$\lim_{t\to\infty} \sup_{\substack{A \in \mathcal{B}[s,t-s] \\ x \in D^+}} \left| P_x\left(Z_\cdot|_{[s,t-s]} \in A \mid \tau_{D^+} > t\right) - P_{\psi^{D^+}}\left(L_\cdot^{D^+}|_{[s,t-s]} \in A\right)\right| = 0. \tag{9}$$

Now let m be an integer which will soon approach infinity, and let $m' = [\sqrt{m}]$, where $[\quad]$ is the greatest integer function. Let $(u_0, v_0) \in \{(x, y) \mid 0 < x < 1, -x+1 \le y < 1\} \setminus D^+$. Put $\Delta_1 = D^+ \cap \{y < v_0\}$ and $\Delta_2 = D^+ \cap \{y > v_0\}$. Let $G(\Delta_1, \Delta_2)$ be the subset of the continuous functions from $[0, 1]$ to $\mathbb{R}^2$ defined by $G(\Delta_1, \Delta_2) = \{f(0) \in \Delta_1, f(1) \in \Delta_2, f(t) \in D^+, 0 \le t \le 1\}$.

Consider the events $F_k = \{L_{k+\cdot}^{D^+}|_{[0,1]} \in G(\Delta_1, \Delta_2)\}$, $k = 0, 1, 2, \ldots$. If $L_0^{D^+}$ has density ψ^{D^+}, then L^{D^+} is a stationary process, and the sequence $I_{F_1}, I_{F_2}, \ldots,$ is stationary. It is easily checked that this sequence is ergodic, using the transition probabilities of L^{D^+} and (3). Let $C_0 = P_{\psi^{D^+}}(F_0)$. The ergodic theorem (see [6]), which says $\lim_{n\to\infty} \sum_{k=0}^{n-1} I_{F_k}/n = C_0$ a.e., gives—here we could replace 4/5 with any constant less than 1—

$$\lim_{n\to\infty} P_{\psi^{D^+}}\left(\frac{\sum_{k=0}^{n-1} I_{F_k}}{n} > \frac{4}{5} C_0\right) = 1, \tag{10}$$

which implies

$$\lim_{m\to\infty} P_{\psi^{D^+}}\left(\frac{\sum_{k=m'}^{m-m'-1} I_{F_k}}{m} > \frac{7}{10} C_0\right) = 1. \tag{11}$$

Using (9), with m and m' in the roles of t and s, (11) gives, for $x \in D^+$,

$$\lim_{m\to\infty} P_x\left(\frac{\sum_{k=m'}^{m-m'-1} I_{\left\{Z_{k+\cdot}|_{[0,1]} \in G(\Delta_1, \Delta_2)\right\}}}{m} > \frac{3}{5} C_0 \mid \tau_{D^+} > m\right) = 1. \tag{12}$$

We now finish the proof of Proposition 2. Recall we are proving Proposition 2 only when $\Gamma = S$ and both $(1,0)$ and $(0,1)$ belong to $\overline{D}$. Let $(u_0, v_0) \in S^+ \setminus D^+$ and satisfy $(u_0, v_0) \in 2^{-q}\mathbb{Z}^2$, for some $q = q(u_0, v_0) \in \mathbb{N}$. This is possible since $\cup_n 2^{-n}\mathbb{Z}^2$ is dense in $\mathbb{R}^2$. (This guarantees the discrete walk, for large enough n, hits the line $y = v_0$ when it crosses from below to above this line.) Let $z_0 = (x_0, y_0) \in D^+$. For integers $n \geq q(u_0, v_0)$, and m, let $A(n, m)$ be the set of all sequences $\boldsymbol{y} = (y_0, y_1, \cdots, y_{\theta_n m})$, satisfying

$$P_{z_0}\left(Y^n_{k\theta_n^{-1}} = y_k, 0 \leq k \leq \theta_n m, \tau^n_{D^+} > m\right) > 0.$$

So certainly $|y_i - y_{i-1}|$ equals either 2^{-n} or zero and $-1 < y_i < 1$ for each i.

Let $\varepsilon > 0$ and let Q be a positive integer. For m a positive integer let $N(n, Q, m, \varepsilon)$ be the subset of $A(n, m)$ consisting of those $\boldsymbol{y}$ such that there exist at least εm integers $h_1 < h_2 < \cdots < h_\alpha$ —so $\alpha \geq \varepsilon m$— such that $h_1 > Q\theta_n$, $h_\alpha < (m - Q)\theta_n$, $|h_i - h_{i-1}| > Q\theta_n$, $2 \leq i \leq \alpha$, and $y_{h_i} = v_0$. Let $m_0 = m_0(Q)$ be the smallest integer such that the probability in (12) exceeds $1/2$ if $m \geq m_0$. If $\boldsymbol{Y}^n_m = (Y^n_{k\theta_n^{-1}})_{1 \leq k \leq m\theta_n}$, we claim that for $m \geq m_0$ and $m' > Q$, there exists $n_1(m)$ such that for $n \geq n_1(m)$,

$$P_{z_0}\left(\boldsymbol{Y}^n_m \in N\left(n, Q, m, \frac{3C_0}{5(Q+2)}\right) \mid \tau^n_{D^+} > m\right) > \frac{1}{2}. \tag{13}$$

To see this, note that

$$\left\{\sum_{k=m'}^{m-m'-1} I_{\{Z_{k+\cdot}|_{[0,1]} \in G(\Delta_1, \Delta_2)\}} > \frac{3C_0 m}{5}, \tau_{D^+} > m\right\}$$

is a union of some of the $2^{m-2m'}$ disjoint sets of the form

$$L_j = \left\{Z_i \in \Delta_{s(i,j)}, m' \leq i \leq m - m' - 1, Z_m \in D^+, \tau_{D^+} > m\right\}$$

where $s(i,j)$ is either 1 or 2. Since each L_j is an event of the form covered by Lemma 5, the definition of m_0 and Lemma 5 show that there exits an $n_1(m)$ such that for $n \geq n_1(m)$,

$$P_x\left(\frac{\sum_{k=m'}^{m-m'-1} I_{\left\{Z^n_{k+\cdot}|_{[0,1]} \in G(\Delta_1, \Delta_2)\right\}}}{m} > \frac{3}{5}C_0 \mid \tau^n_{D^+} > m\right) > \frac{1}{2}. \tag{14}$$

Now if a path of Z^n takes values in Δ_1 at i and in Δ_2 at time $i+1$ then at some time $l(i)\theta_n^{-1}$ between these two times the y-coordinate of the path of Z^n must equal v_0. Furthermore of a collection of η such times, each corresponding to a different integer i, at least $\eta/(Q+2)$ may be chosen which are all a distance Q from all the others: let the smallest be the first chosen, the $Q+2$ smallest be the second chosen, the $2(Q+2)$ smallest be the third chosen, and so on. Therefore, (14) and the above observations imply (13).

Let $P^{\boldsymbol{y}}$ designate conditional probability associated with $Z^n_0, \ldots, Z^n_m$ given $Y^n_{i\theta_n^{-1}} = y_i$, $1 \leq i \leq m\theta_n$.

Lemma 6 *There is an integer $K_0 = K_0(u_0) > 0$ and a number $0 < d = d(u_0) < 1$ such that if $\boldsymbol{y} \in A(n,m)$ and if there are λ entries $y_{j_1}, y_{j_2}, \dots, y_{j_\lambda}$ of $\boldsymbol{y}$ which satisfy $|j_i - j_{i-1}| > K_0\theta_n$, $K_0\theta_n < j_1$, and $j_\lambda < (m - K_0)\theta_n$ and $y_{j_i} = v_0$, $1 \le i \le \lambda$, then there is an integer $n_2 = n_2(m)$ depending only on m but not on $\boldsymbol{y}$ such that*

$$\frac{P^{\boldsymbol{y}}_{z_0}\left(\tau^n_{D+} > m\right)}{P^{\boldsymbol{y}}_{z_0}\left(\tau^n_{D} > m\right)} \le d^{\lambda} \frac{P_{z_0}\left(\tau^n_{S+} > m\right)}{P_{z_0}\left(\tau^n_{S} > m\right)}, \quad n \ge n_2(m). \tag{15}$$

Before proving Lemma 6 we show how it implies Proposition 2. Let $\delta_0 = 3C_0/(5(K_0+2))$. Now (15) and the definition of $N(n, K_0, m, \delta_0)$ yield

$$\begin{aligned} &P^{\boldsymbol{y}}_{z_0}\left(\tau^n_{D+} > m\right) P_{z_0}\left(\boldsymbol{Y}^n_m = \boldsymbol{y}\right) \\ &\qquad \le d^{\delta_0 m} \frac{P_{z_0}\left(\tau^n_{S+} > m\right)}{P_{z_0}\left(\tau^n_{S} > m\right)} \times P^{\boldsymbol{y}}_{z_0}(\tau^n_D > m) P_{z_0}(\boldsymbol{Y}^n_m = \boldsymbol{y}), \quad n \ge n_2(m), \end{aligned}$$

for $\boldsymbol{y} \in N(n, K_0, m, \delta_0)$. Summing these over all $\boldsymbol{y}$ in $N(n, K_0, m, \delta_0)$ gives

$$\begin{aligned} &\frac{P_{z_0}\left(\tau^n_{D+} > m, \boldsymbol{Y}^n_m \in N(n, K_0, m, \delta_0)\right)}{P_{z_0}\left(\tau^n_{D} > m, \boldsymbol{Y}^n_m \in N(n, K_0, m, \delta_0)\right)} \\ &\le d^{\delta_0 m} \frac{P_{z_0}\left(\tau^n_{S+} > m\right)}{P_{z_0}\left(\tau^n_{S} > m\right)}, \qquad n \ge n_2(m). \end{aligned} \tag{16}$$

Now (13) implies that for $m \ge m_0(K_0)$, $m' > K_0$, and $n \ge n_1(m)$, the numerator in the left side of (16) is at least half of $P_{z_0}(\tau^n_{D+} > m)$, while of course the denominator on the left of (16) is no larger than $P_{z_0}(\tau^n_D > m)$. Thus (16) gives

$$\frac{P_{z_0}\left(\tau^n_{D+} > m\right)}{P_{z_0}\left(\tau^n_{D} > m\right)} \le 2d^{\delta_0 m} \frac{P_{z_0}\left(\tau^n_{S+} > m\right)}{P_{z_0}\left(\tau^n_{S} > m\right)}, \tag{17}$$

if $m \ge m_0$, $m' > K_0$, and $n \ge n_1(m)$.

Letting $n \to \infty$ for fixed m and using (7) gives Proposition 2 in the case $\Gamma = S$ and t an integer and $o(1) \le 2d^{\delta_0 t}$, which easily implies Proposition 2 for all t. We note that, by reasoning similar to that of the paragraph containing (2), this bound on $o(1)$ implies $\lambda^{D^+} - \lambda^D - \delta_0(\log d) \ge \lambda^{S^+} - \lambda^S$.

The proof of Lemma 6 requires Lemmas 7, 8, and 9 below. Lemma 7 is from [9]. See [10] for an easier proof.

Lemma 7 *Let $m > 0$ be a positive integer, and let f and g be integer valued functions on $\{0, 1, 2, \dots, m\}$ such that $2 \le f(k) \le g(k)$, $0 \le k \le m$. Let $R_0, R_1, \dots$ be random walk as defined at the beginning of this section and let i_0 be an integer in $(0, f(0))$. Then*

$$\frac{P_{i_0}\left(0 < R_k < f(k), 0 \le k \le m\right)}{P_{i_0}\left(0 < |R_k| < f(k), 0 \le k \le m\right)} \le \frac{P_{i_0}\left(0 < R_k < g(k), 0 \le k \le m\right)}{P_{i_0}\left(|R_k| < g(k), 0 \le k \le m\right)}. \tag{18}$$

The eigenfunctions of the intervals $(0,1)$ and $(-1,1)$ are $\phi^{(0,1)}(x) = \frac{\pi}{2}\sin \pi x$ and $\phi^{(-1,1)}(x) = \frac{\pi}{4}\cos(\frac{\pi}{2}x)$. For $0 < \alpha < 1$, let

$$\beta(\alpha) = \frac{\int_0^\alpha \phi^{(0,1)}(x)^2\,dx}{\int_0^1 \phi^{(0,1)}(x)^2\,dx}$$

and

$$\gamma(\alpha) = \frac{\int_{\{|x|<\alpha\}} \phi^{(-1,1)}(x)^2\,dx}{\int_{-1}^1 \phi^{(-1,1)}(x)^2\,dx}.$$

Then since $\phi^{(0,1)}$ and $\phi^{(-1,1)}$ have the same shape (or by direct calculation) we see that $\beta(\alpha) < \gamma(\alpha)$.

Lemma 8 *Let $0 < \alpha < 1$ and let $\varepsilon > 0$, and put $\beta = \beta(\alpha)$ and $\gamma = \gamma(\alpha)$. There is a number $K = K(\alpha, \varepsilon)$ such that, for any integer m, if $t_1, t_2, \dots, t_m$, and T are numbers such that*

$$K < t_1 < t_2 < \cdots < t_m < T - K \text{ satisfy } |t_i - t_{i-1}| \geq K, \tag{19}$$

and $w \in (0,1)$, then

$$P_w\left(W_{t_i} < \alpha, 1 \leq i \leq m \mid \tau_{(0,1)} > T\right) \leq (\beta + \varepsilon)^m, \tag{20}$$

and

$$P_w\left(|W_{t_i}| < \alpha, 1 \leq i \leq m \mid \tau_{(-1,1)} > T\right) \geq (\gamma - \varepsilon)^m. \tag{21}$$

Proof We prove (20). The proof of (21) is similar. If $0 < s < t$ and $x \in (0,1)$, the joint density $\eta(y,z)$ of (W_s, W_t), conditioned on $\tau_{(0,1)} > t$ and $W_0 = x$, is the normalization of $p_s^{(0,1)}(x,y)p_{t-s}^{(0,1)}(y,z)$. Thus, the density $h_{x,z}$ of W_s given $W_0 = x$ and $W_t = z$, and $\tau_{(0,1)} > t$ is

$$h_{x,z}(y) = \frac{p_s^{(0,1)}(x,y)p_{t-s}^{(0,1)}(y,z)}{\int_0^1 p_s^{(0,1)}(x,y)p_{t-s}^{(0,1)}(y,z)\,dy}.$$

The one dimensional version of (3) and the fact that $C^{(0,1)}(v)$ decreases to 1 and $c^{(0,1)}(v)$ increases to 1 as v increases imply that if $v = \min(s, t-s)$, then

for all z and x in $(0,1)$

$$
\begin{aligned}
&\frac{e^{-\lambda^{(0,1)}t}c^{(0,1)}(v)^2\phi^{(0,1)}(x)\phi^{(0,1)}(y)^2\phi^{(0,1)}(z)}{\left(\int_0^1 \phi^{(0,1)}(x)^2\,dx\right)^2} \qquad (22)\\
&\le \frac{e^{-\lambda^{(0,1)}s}e^{-\lambda^{(0,1)}(t-s)}c^{(0,1)}(s)c^{(0,1)}(t-s)\phi^{(0,1)}(x)\phi^{(0,1)}(y)^2\phi^{(0,1)}(z)}{\left(\int_0^1 \phi^{(0,1)}(x)^2\,dx\right)^2}\\
&\le p_s^{(0,1)}(x,y)p_{t-s}^{(0,1)}(y,z)\\
&\le \frac{e^{-\lambda^{(0,1)}s}e^{-\lambda^{(0,1)}(t-s)}C^{(0,1)}(s)C^{(0,1)}(t-s)\phi^{(0,1)}(x)\phi^{(0,1)}(y)^2\phi^{(0,1)}(z)}{\left(\int_0^1 \phi^{(0,1)}(x)^2\,dx\right)^2}\\
&\le \frac{e^{-\lambda^{(0,1)}t}C^{(0,1)}(v)^2\phi^{(0,1)}(x)\phi^{(0,1)}(y)^2\phi^{(0,1)}(z)}{\left(\int_0^1 \phi^{(0,1)}(x)^2\,dx\right)^2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{\phi^{(0,1)}(x)\phi^{(0,1)}(z)e^{-\lambda^{(0,1)}t}c^{(0,1)}(v)^2}{\int_0^1 \phi^{(0,1)}(x)^2\,dx} &\le \int_0^1 p_s^{(0,1)}(x,y)p_{t-s}^{(0,1)}(y,z)\,dy \qquad (23)\\
&\le \frac{\phi^{(0,1)}(x)\phi^{(0,1)}(z)e^{-\lambda^{(0,1)}t}C^{(0,1)}(v)^2}{\int_0^1 \phi^{(0,1)}(x)^2\,dx}.
\end{aligned}
$$

Together, (22) and (23) imply that

$$\frac{c^{(0,1)}(v)^2}{C^{(0,1)}(v)^2}\psi^{(0,1)}(y) < h_{x,z}(y) < \frac{C^{(0,1)}(v)^2}{c^{(0,1)}(v)^2}\psi^{(0,1)}(y).$$

This implies that given $\varepsilon > 0$ there is $Q_{(0,1)} = Q_{(0,1)}(\alpha,\varepsilon)$ such that for $v \ge Q_{(0,1)}/2$ and all $x,z \in (0,1)$,

$$\int_0^\alpha h_{x,z}(y)\,dy < \beta + \varepsilon. \tag{24}$$

Let $\hat{P}$ be conditional probability given $\tau_{(0,1)} > T$. Then under $\hat{P}$, W_t is a Markov process, although not with stationary transition probabilities. Let $A_i = \{W_{t_i} < \alpha\}$. Let $t_m^+ = T$, and for $1 \le i < m$ let $t_i^+ = (t_i + t_{i+1})/2$. Suppose T and t_i, $1 \le i \le m$, satisfy (19) with $Q_{(0,1)}$ in place of K. Then (20) holds by the following argument, which can be made rigorous by changing $W_{t_i} = z_i$ to

$W_{t_i} \in [z_i, z_i + dz]$.

$$\begin{aligned}
\hat{P}_w\left(A_1 \mid W_{t_1^+} = z_1\right) &= \frac{\hat{P}_w\left(A_1, W_{t_1^+} = z_1\right)}{\hat{P}_w\left(W_{t_1^+} = z_1\right)} \\
&= \frac{P_w\left(A_1, W_{t_1^+} = z_1, \tau_{(0,1)} > t_m^+\right)}{P_w\left(W_{t_1^+} = z_1, \tau_{(0,1)} > t_m^+\right)} \\
&= \frac{P_w\left(A_1, W_{t_1^+} = z_1, \tau_{(0,1)} > t_1^+\right) P_{z_1}\left(\tau_{(0,1)} > t_m^+ - t_1^+\right)}{P_w\left(W_{t_1^+} = z_1, \tau_{(0,1)} > t_1^+\right) P_{z_1}\left(\tau_{(0,1)} > t_m^+ - t_1^+\right)} \\
&< \beta + \varepsilon \qquad \text{by (24).}
\end{aligned}$$

Using similar ratios and the Markov property we obtain $\hat{P}_w(A_2 \mid A_1, W_{t_1^+} = z_1, W_{t_2^+} = z_2) = \hat{P}_{z_1}(A_1 \mid W_{t_1^+} = z_2) < \beta + \varepsilon$ so that $\hat{P}_w(A_1 \cap A_2 \mid W_{t_2^+} = z_m) < (\beta + \varepsilon)^2$. Proceeding in this manner gives

$$\hat{P}_w\left(A_1 \cap A_2 \cap \cdots \cap A_m \mid W_{t_m^+} = z_m\right) < (\beta + \varepsilon)^m,$$

and integrating over z_m gives (20). Similarly there is a $Q_{(-1,1)}(\alpha, \varepsilon)$ such that if T and t_i, $1 \le i \le m$, satisfy (19) with $Q_{(-1,1)}$ in place of K, then (21) holds. So $K(\alpha, \varepsilon)$ can be and is taken to be the smallest integer larger than $\max(Q_{(0,1)}(\alpha, \varepsilon), Q_{(-1,1)}(\alpha, \varepsilon))$.

Let $d = d(u_0) = (\frac{\beta(u_0)}{\gamma(u_0)} + 1)/2 < 1$, and let $\varepsilon = \varepsilon(u_0)$ satisfy $\frac{\beta+\varepsilon}{\gamma-\varepsilon} < \frac{1}{2}(d + \frac{\beta(u_0)}{\gamma(u_0)})$. Let $K_0 = K(u_0, \varepsilon(u_0))$. Lemma 8 implies

$$\begin{aligned}
&\frac{P_w\left(W_{t_i} < u_0, 1 \le i \le m, \text{ and } \tau_{(0,1)} > T\right)}{P_w\left(|W_{t_i}| < u_0, 1 \le i \le m, \text{ and } \tau_{(0,1)} > T\right)} \\
&< d^m \frac{P_w\left(\tau_{(0,1)} > T\right)}{P_w\left(\tau_{(-1,1)} > T\right)}.
\end{aligned} \tag{25}$$

Lemma 9 *Let T, $t_1, t_2, \ldots, t_m$ be as in Lemma 8, and suppose in addition that all t_i, $1 \le i \le m$, are in $\Theta(n)$ for some n and $w = l2^{-n}$ where l is an integer such that $0 < l < 2^n$. Then there is an integer $N(m, T)$ such that*

$$\begin{aligned}
&\frac{P_w\left(W_{t_i}^{(n)} < u_0, 1 \le i \le m, \textit{ and } \tau_{(0,1)}^{(n)} > T\right)}{P_w\left(|W_{t_i}^{(n)}| < u_0, 1 \le i \le m, \textit{ and } \tau_{(0,1)}^{(n)} > T\right)} \\
&< d^m \frac{P_w\left(\tau_{(0,1)}^{(n)} > T\right)}{P_w\left(\tau_{(-1,1)}^{(n)} > T\right)} \quad n \ge N(m, T).
\end{aligned} \tag{26}$$

Proof That the limit as $n \to \infty$ of both of the numerators and both of the denominators in (26) exists follows from a one dimensional version of Lemma 5, which implies that these limits are the analogous probabilities for Brownian motion W_t. This one dimensional version follows from the classical result of Skorohod that processes with the distribution of $W^{(n)}$ may be embedded in W in such a way that given $t > 0$ and $\varepsilon > 0$, $P(|W_s - W_s^{(n)}| < \varepsilon, 0 \le s \le t) > 1 - \varepsilon$ for large enough n. Together with (25) this establishes Lemma 9.

We note that, for $z_0 = (x_0, y_0)$, the independence of the components X^n and Y^n of Z^n implies that if m is an integer both

$$\begin{aligned} P_{z_0}\left(\tau^n_{S+} > m\right) = & P_{x_0}\left(\inf\left\{t \mid W^n_t \notin (0,1)\right\} > m\right) \\ & \times P_{y_0}\left(\inf\left\{t \mid W^n_t \notin (-1,1)\right\} > m\right) \end{aligned}$$

and

$$\begin{aligned} P_{z_0}\left(\tau^n_{S} > m\right) = & P_{x_0}\left(\inf\left\{t \mid W^n_t \notin (-1,1)\right\} > m\right) \\ & \times P_{y_0}\left(\inf\left\{t \mid W^n_t \notin (-1,1)\right\} > m\right) \end{aligned}$$

where W^n_t is the one dimensional random walk defined at the beginning of this section. Thus the ratio of the right hand side of (15) is

$$\frac{P_{2^n x_0}\left(R_k \in (0, 2^n), 0 \le k \le m\theta_n\right)}{P_{2^n x_0}\left(R_k \in (-2^n, 2^n), 0 \le k \le m\theta_n\right)} = \frac{P_{x_0}\left(\tau^n_{(0,1)} > m\right)}{P_{x_0}\left(\tau^n_{(-1,1)} > m\right)}. \tag{27}$$

Now we finish the proof of Lemma 6. We think of $\boldsymbol{y}$ as fixed. Let $q_n(k) = \max\{i \in \mathbb{Z} \mid (i2^{-n}, y_k) \in D\}$, $1 \le k \le m2^n$, and $Q(n) = \max\{i \in \mathbb{Z} \mid (i2^{-n}, v_0) \in D\}$, and put $\hat{q}_n(k) = Q(n)$ for $k = j_i$, $1 \le i \le \lambda$, $\hat{q}_n(k) = 2^n$ for all other $k \in \mathbb{N}$. Note $q_n(k) \le \hat{q}_n(k)$, $0 \le k \le m\theta_n$.

Then using Lemma 7 we get

$$\frac{P_{2^n x_0}\left(0 < R_k < q(k), 1 \le k \le m\theta_n\right)}{P_{2^n x_0}\left(|R_k| < q(k), 1 \le k \le m\theta_n\right)} \le \frac{P_{2^n x_0}\left(0 < R_k < \hat{q}(k), 1 \le k \le m\theta_n\right)}{P_{2^n x_0}\left(|R_k| < \hat{q}(k), 1 \le k \le m\theta_n\right)}. \tag{28}$$

Now, upon scaling, the left hand ratio in (28) becomes

$$\frac{P^{\boldsymbol{y}}_{x_0}(\tau^n_{D+} > m)}{P^{\boldsymbol{y}}_{x_0}(\tau^n_{D} > m)},$$

which is the left hand side of (15), while the ratio on the right hand side of (28) becomes

$$\frac{P_{x_0}\left(0 < W^n_{j_i\theta_n^{-1}} < u_0, 1 \le i \le \lambda, \tau^n_{(0,1)} > m\right)}{P_{x_0}\left(|W^n_{j_i\theta_n^{-1}}| < u_0, 1 \le i \le \lambda, \tau^n_{(-1,1)} > m\right)},$$

which by Lemma 9 does not exceed

$$d^\lambda \frac{P_{x_0}(\tau^n_{(0,1)} > m)}{P_{x_0}(\tau^n_{(-1,1)} > m)},$$

which is the right hand of (15). This proves (15).

3 Proof of Theorem 3

Again we just prove Theorem 3 in the case $\Gamma = S$ and when $(0,1)$ and $(1,0)$ are in $\overline{D}$ and we let (u_0, v_0) be a point in $(S \setminus D) \cap \{x > 0, y > 0\}$. Again the argument for arbitrary a and b is virtually identical.

In Section 2, just after (17), we showed that

$$\lambda^{D^+} - \lambda^D - \delta_0 \log d \geq \lambda^{S^+} - \lambda^S. \tag{29}$$

Now $d = d(u_0)$ is explicitly defined in the last section. To complete the proof of Theorem 3 we show that there is a computable positive number $h(u_0, v_0)$ depending only on u_0 and v_0 alone for which the inequality that results when that number is substituted for δ_0 in (29) is true. In other words,

$$\lambda^{D^+} - \lambda^D - h(u_0, v_0) \log d(u_0) \geq \lambda^{S^+} - \lambda^S, \tag{30}$$

and Theorem 3 in the case $\Gamma = S$ is verified, where

$$g(u,v) = \min(h(u,v) \log d(u), h(v,u) \log d(v)).$$

Here we use the fact that if (u_0, v_0) is omitted from D^+ then (v_0, u_0) is omitted from the 90° rotation of D^T.

We need to show how to produce a positive function of u_0 and v_0 which is smaller than δ_0. Recall $\delta_0 = 3P_{\psi^{D^+}}(F_0)/(5(K_0 + 2))$. Now the proof of Lemma 10 will show how to bound $P_{\psi^{D^+}}(F_0)$ below. Also, $K_0 = K(u_0, \varepsilon(u_0))$, and $\varepsilon(u_0)$ is easily computed while K_0 is defined just above (25). The proof of Lemma 8 shows that if we can produce explicit versions of the functions $c^{(0,1)}(v)$ and $C^{(0,1)}(v)$ then the desired explicit upper bound for K_0 can be achieved. These explicit versions can be found using either the exact formulas for the one-dimensional heat kernels or the exact formulas of the eigenvalues of an interval.

Lemma 10 *The quantity $P_{\psi^{D^+}}(F_0)$ may be bounded below by a positive number which depends only on u_0 and v_0.*

We first prove some estimates for $p_1^{D^+}(z,w)$ and ϕ^{D^+}. For $z \in D^+$, let $\eta(z)$ be the distance from z to the boundary of D^+, and let D^+_ε be all points z of D^+ which satisfy $\eta(z) > \varepsilon$. For z, $w \in D^+$, let $R(z,w)$ be the rectangle which lies in D^+, has one side of length $\min(\eta(z), \eta(w))$, and contains z and w and such that both z and w are a distance $\min(\eta(z), \eta(w))/2$ from three of the four sides of $R(z,w)$. (So, if either z or w are close to the boundary of D^+ and z and w are far apart, $R(z,w)$ is a long skinny rectangle and z and w are both close to short sides of $R(z,w)$.) The convexity of D^+ guarantees $R(z,w) \subset D^+$. The heat kernel of a rectangle is exactly known and using this exact formula and the fact that the diameter of D^+ does not exceed 3, so the long side of $R(z,w)$ is no longer than 3, it is easy to give the equation of a positive increasing function $\beta(s)$, $s > 0$ such that

$$p_1^{R(z,w)}(z,w) \geq \beta(\min(\eta(z), \eta(w))), \quad z, w \in D^+,$$

which implies

$$p_1^{D^+}(z,w) \geq \beta(\min(\eta(z),\eta(w))), \quad z,w \in D^+. \tag{31}$$

Also, it is immediate that

$$p_1^{D^+}(z,w) \leq p_1^{\mathbb{R}^2}(z,w) = \frac{1}{2\pi}. \tag{32}$$

Now since the rectangle $(0,1/2) \times (-1/2,1/2)$, which has first eigenvalue $3\pi^2$, is contained in D^+, we know

$$\lambda^{D^+} \leq 3\pi^2. \tag{33}$$

Using (32) and (33) we get

$$e^{-3\pi^2}\phi^{D^+}(z) \leq e^{-\lambda^{D^+}}\phi^{D^+}(z) = \int_{D^+} \phi^{D^+}(x) p_1^{D^+}(x,z)\,dz \leq \frac{1}{2\pi}, \quad z \in D^+,$$

yielding

$$\phi^{D^+}(z) \leq \frac{e^{3\pi^2}}{2\pi}, \quad z \in D^+. \tag{34}$$

Also,

$$\begin{aligned} \int_{D^+} \phi^{D^+}(z)^2\,dz &= \text{ area } D^+ \int_{D^+} \phi^{D^+}(z)^2\, d\left(\frac{z}{\text{area } D^+}\right) \\ &\geq \text{ area } D^+ \left[\int_{D^+} \phi^{D^+}(z)\, d\left(\frac{z}{\text{area } D^+}\right)\right]^2 \\ &= (\text{area } D^+)\cdot(\text{area } D^+)^{-2} \geq \frac{1}{2}. \end{aligned} \tag{35}$$

Now if $z \in D^+ \setminus D^+_\varepsilon$, the convexity of D^+ implies that either there is a point on the vertical line through z which belongs to the boundary of D^+ and is a distance at most $\sqrt{2}\varepsilon$ from z or a point on the horizontal line through z which belongs to the boundary of D^+ and is a distance at most $\sqrt{2}\varepsilon$ from z. Let $L^x = \{(x,y) \mid -\infty < y < \infty\}$ and $L_y = \{(x,y) \mid -\infty < x < \infty\}$. Let ∂D stand for the boundary of D. Then for almost every x in $(0,1)$ and every y in $(-1,1)$, $L^x \cap \partial D^+$ and $L_y \cap \partial D^+$ each consist of exactly two points. Let $L^x(a)$ be all points w in D^+ which belong to L^x and such that there is $z \in L^x \setminus D^+$ such that $|z-w| \leq a$. Let $L_y(a)$ be all points w in D^+ that belong to L_y and such that there is a point z in $L_y \setminus D^+$ such that $|z-w| \leq a$. Then the length of $L_y(a)$ does not exceed $2a$ when $L_y \cap \partial D^+$ consists of two points, with a similar inequality for $L^x(a)$. Since

$$\bigcup_{0\leq x\leq 1} L^x(\sqrt{2}\varepsilon) \cup \bigcup_{-1\leq y\leq 1} L_y(\sqrt{2}\varepsilon) \supset D^+ \setminus D^+_\varepsilon,$$

$$|D^+ \setminus D^+_\varepsilon|_2 \leq \int_0^1 |L^x(\sqrt{2}\varepsilon)|_1\,dx + \int_{-1}^1 |L_y(\sqrt{2}\varepsilon)|_1\,dy \leq 3\cdot 2\sqrt{2}\varepsilon, \tag{36}$$

where $|\quad|$ is respectively Lebesgue two and one dimensional measure.

The inequalities (34), (35), and (36) give

$$\int_{D^+\setminus D^+_\varepsilon} \psi^{D^+} \le 6\sqrt{2}\varepsilon \left(\frac{e^{3\pi^2}}{2\pi}\right)^2 \cdot 2. \tag{37}$$

Thus, since $\int_{D^+} \psi^{D^+}(z)\,dz = 1$, we have that if $\gamma = [6\sqrt{2}(e^{3\pi^2}/2\pi)^2 \cdot 2]^{-1} \cdot 1/2$, then $\int_{D^+\setminus D^+_\gamma} \psi^{D^+} \le 1/2$, which, together with the symmetry about the x-axis of ϕ^{D^+} and thus ψ^{D^+}, and the fact that $\int_{D^+} \psi^{D^+} = 1$ gives

$$\int_{D^+_\gamma \cap \{y<0\}} \psi^{D^+} \ge \frac{1}{4}. \tag{38}$$

Furthermore, by (34) and (36),

$$\int_{D^+_\gamma} \phi^{D^+}(x)\,dx \ge \frac{1}{4}. \tag{39}$$

We also note that, by (31) and (39),

$$\begin{aligned} \phi^{D^+}(z) &= e^{\lambda^{D^+}} \int_{D^+} \phi^{D^+}(x) p_1^{D^+}(x,z)\,dx \qquad (40)\\ &\ge \int_{D^+} \phi^{D^+}(x) p_1^{D^+}(x,z)\,dx \\ &\ge \int_{D^+_\gamma} \phi^{D^+}(x) p_1^{D^+}(x,z)\,dx \\ &\ge \frac{1}{4}\beta(\min(\eta(z),\gamma)). \end{aligned}$$

We now bound $P_{\psi^{D^+}}(F_0)$ from below. Using the transition probabilities for $L_t^{D^+}$ given by (8), we have

$$P_{\psi^{D^+}}(F_0) = \int_{D^+\cap\{y<v_0\}} \left[\int_{D^+\cap\{y>v_0\}} l_1^{D^+}(x,y)\,dy\right] \psi^{\Omega}(x)\,dx.$$

Now the open triangle $T(v_0)$ with vertices $(0,1)$, $(0,v_0)$, and $(1-v_0,v_0)$ must lie in $D^+\cap\{y>v_0\}$. Let $t(v_0)$ be the middle third of this open triangle, that is, $t(v_0)$ is the translation of $\frac{1}{3}T(v_0)$ satisfying that the medians of $t(v_0)$ and $T(v_0)$ meet in the same place. Then all points of $t(v_0)$ are at least a distance

$v_0/3$ from ∂D^+, and so since the area of $t(v_0)$ equals $v_0^2/18$,

$$
\begin{aligned}
P_{\psi^{D^+}}(F_0) &\geq \int_{D^+_\gamma \cap \{y<v_0\}} \left[\int_{t(v_0)} l_1^{D^+}(x,y)\, dy \right] \psi^{D^+}(x)\, dx \\
&\geq \int_{D^+_\gamma \cap \{y<v_0\}} \left[\int_{t(v_0)} p_1^{D^+}(x,y) \frac{2\pi\beta(\min(\gamma, \frac{v_0}{3}))}{4e^{3\pi^2}}\, dy \right] \psi^{D^+}(x)\, dx \\
&\geq \frac{2\pi\beta(\min(\gamma, \frac{v_0}{3}))}{4e^{3\pi^2}} \int_{D^+_\gamma \cap \{y<0\}} \left[\int_{t(v_0)} p_1^{D^+}(x,y)\, dy \right] \psi^{D^+}(x)\, dx \\
&\geq \frac{2\pi\beta(\min(\gamma, \frac{v_0}{3}))}{4e^{3\pi^2}} \int_{D^+_\gamma \cap \{y<0\}} \beta\left(\min(\gamma, \frac{v_0}{3})\right) \frac{v_0^2}{18} \psi^{D^+}(x)\, dx \\
&\geq \frac{2\pi\beta(\min(\gamma, \frac{v_0}{3}))^2}{4e^{3\pi^2}} \cdot \frac{v_0^2}{18} \cdot \frac{1}{4},
\end{aligned}
$$

using (34) and (40) in the second inequality, (31) in the fourth, and (38) in the last. This proves Lemma 10, which completes the proof of Theorem 3.

The existence of a positive function $g(u,v)$ which satisfies the statement that results if "computable" is removed from the statement of Theorem 3 follows from Theorem 1. For this existence is equivalent to the statement that the supremum of the set of all gaps of convex doubly symmetric domains contained in Γ with closures containing $(a,0)$ and $(0,b)$ which do not contain (u_0,v_0) is less than the gap of Γ. Suppose by way of contradiction that this is not the case. Pick a sequence of these domains such that their gaps converge to the gap of Γ. Pick a subsequence D_n of these domains such that $\sup\{y \mid (x,y) \in D_n\}$ converges, say to $f(x)$, for each rational x. Let D' be the unique doubly symmetric convex domain such that $\sup\{(x,y) \mid y \in D'\} = f(x)$ for each rational x in $(-1,1)$. The monotonicity property of the eigenvalues (see Theorem VI.3 in [7]) together with the fact that for $\varepsilon > 0$ if n is large enough then $(1-\varepsilon)D_n \subset D' \subset (1+\varepsilon)D_n$ implies that the gap of D' equals the gap of Γ. And (u_0,v_0) does not belong to D', yielding a contradiction to Theorem 1.

4 Final Comments

In this section we drop our convention that D always stands for a convex domain, and discuss some possible extensions of the results we have proved. We believe that our proof of Proposition 2 will extend fairly easily to prove a stronger version of Proposition 2 in which the convexity and double symmetry of D is weakened to convexity in x and symmetry about the y-axis. Under this weakened condition D is no longer intrinsically ultracontractive. However the analog of (9), the only place we used two dimensional intrinsic ultracontractivity, actually holds for all bounded domains.

In [9] and [2] a higher dimensional analog is proved of the two-dimensional result of [9] that $\lambda^{D^+} - \lambda^D \geq \lambda^{\Gamma^+} - \lambda^\Gamma$ if D is simply connected, bounded, symmetric about the y-axis, convex in x, and contained in $(-a,a) \times (-b,b)$.

We believe that properly formulated analogs of Proposition 2 are true, but note that, as heat kernels do not see a line in higher dimensions, care needs to be taken in the formulation.

In [3] and later in [2], [10] and [11], theorems related to the results of [9] were proved for Schrödinger operators with potential kernels which are symmetric about the y-axis and nondecreasing in x for positive x for fixed y and which are defined on bounded domains symmetric about the y-axis. We believe that the methods in [10] and those of this paper can be used to prove analogs of Proposition 2 of this paper for such operators.

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